Differential Equations : Solutions

David W.H. Swenson

Exercise 1. If the general solution to a differential equation is $y(x) = Ax^2 + Bx + C$ and it is subject to the initial conditions $y(0) = 5$, $y'(0) = 2$, and $y''(0) = 1$, then what are A, B, and C?

We just need to apply each of the boundary conditions. For the boundary condition $y(0) = 5$ we have

$$A(0)^2 + B(0) + C = C = 5$$

Since $y'(x) = 2Ax + B$, the second initial condition gives us

$$2A(0) + B = B = 2$$

Finally, $y''(x) = 2A$, so the last initial condition gives

$$2A = 1 \implies A = \frac{1}{2}$$

Exercise 2. Let's assume that we've already found that the general solution to a differential equation is $y(x) = A\cos(\lambda x) + B\sin(\lambda x)$. Now we will subject this system to several boundary conditions:

1. If the system is constrained by the boundary condition $y(0) = 0$, which variable is fixed, and what does $y(x)$ become?

2. Using your the new form of $y(x)$, which variable is fixed by the boundary condition $y(L) = 0$ for an arbitrary L? [Hint : you *don't* want to end up with the "trivial" solution of $y(x) = 0$.] What is now the new solution for $y(x)$?

3. Now think carefully about your second answer. Did you account for every possible value for which $y(L) = 0$? Since our initial function is an oscillatory function, there will be an infinite number of these. Introduce an integer variable n such that you account for all the possibilities. Write down the new form of $y(x)$.

4. You should only have one parameter still undetermined from the original general solution. Let's fix that parameter by requiring that it satisfy the boundary condition $\int_0^L \mathrm{d}x\, y^2(x) = 1$. What is your end result for $y(x)$?

[Hint: take another look at this exercise when you deal with the particle in a box.]

By difficulty, this should be an Advanced exercise, but it is so fundamentally important to quantum mechanics that I hope everyone will work through it.

Let's follow the order in the problem:

1. With the boundary condition $y(0) = 0$, we have

$$\begin{aligned} 0 &= A\cos(0) + B\sin(0) \\ &= A \end{aligned}$$

so the parameter A is fixed to be zero, and the new equation is $y(x) = B\sin(\lambda x)$.

2. Now, with $y(x) = B\sin(\lambda x)$, we want to fix $y(L) = 0$. That is, we want to find the solution such that

$$0 = B\sin(\lambda L)$$

There are two ways we could do this: if we set $B = 0$, the problem is solved. Unfortunately, that also gives us the trivial solution $y(x) = 0$ which is generally not going to be the physical solution. So instead we want to find λ such that $\sin(\lambda L) = 0$. Again, we don't want $\lambda = 0$, because that would give us the trivial solution. So we take $\lambda = \pi/L$, which will clearly work since $\sin(\pi) = 0$. In this part, we fixed λ, and a new solution is $y(x) = B\sin(\pi x/L)$.

3. What we did in part 2 clearly gives us *a* solution to the problem, but is it the only solution? If $\lambda = \pi/L$ works, why not $\lambda = 2\pi/L$? Or $\lambda = 3\pi/L$? In reality, all of these work, and so we should really put in a positive integer n and say that $\lambda = n\pi/L$. Although this may seem like an unimportant mathematical detail, we'll see in exercise 9 that *this is reason energy levels are quantized* in the particle in a box. Make sure that makes sense to you: it's really important. Anyway, the new solution after all of this is $y(x) = B\sin(n\pi x/L)$.

4. One variable left, one boundary condition to satisfy. We'll be fixing B by solving

$$1 = \int_0^L \mathrm{d}x\, B^2 \sin^2\left(\frac{n\pi}{L}x\right)$$

The reasonable folks will look that up in a table of integrals. The less reasonable folks (like me) will solve it by hand. To do that, let's first make the substitution $u = \frac{n\pi}{L}x$, which also gives us $\mathrm{d}x = \frac{L}{n\pi}\mathrm{d}u$ and changes the integral to:

$$
\begin{aligned}
1 &= B^2 \frac{L}{n\pi} \int_0^{n\pi} \mathrm{d}u \, \sin^2(u) = B^2 \frac{L}{n\pi} \int_0^{n\pi} \mathrm{d}u \, \sin(u) \sin(u) \\
&= B^2 \frac{L}{n\pi} \int_0^{n\pi} \mathrm{d}u \, \frac{1}{2} \left(\cos(u-u) - \cos(u+u) \right) \\
&= B^2 \frac{L}{n\pi} \int_0^{n\pi} \mathrm{d}u \, \frac{1}{2} \left(1 - \cos(2u) \right) \\
&= B^2 \frac{L}{n\pi} \frac{1}{2} \left(\int_0^{n\pi} \mathrm{d}u \, - \int_0^{n\pi} \mathrm{d}u \, \cos(2u) \right) \\
&= B^2 \frac{L}{2n\pi} \left(n\pi + \left(\frac{1}{2} \sin(2u) \right) \Big]_0^{n\pi} \right) \\
&= B^2 \frac{L}{2n\pi} \left(n\pi + 0 - 0 \right) = B^2 \frac{L}{2}
\end{aligned}
$$

That integral can also be done by converting to complex numbers, or by noting that the integral is over n periods, comparing the integral over of a period of $\sin^2(\theta)$ with that of $\cos^2(\theta)$, and by taking the integral of the relation $\sin^2(\theta) + \cos^2(\theta) = 1$. I'll leave the details of that method to the curious.

However you get it, the result is that $B = \sqrt{\frac{2}{L}}$. This gives us a final solution (wavefunction of the particle in a box) of $y(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right)$.

Exercise 3. Confirm that $y = c\, e^{ax}$ satisfies the differential equation $y' = ay$. That is, plug the solution function into the differential equation and show that it works.

This is just a matter of sticking the solution into the differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(c\, e^{ax} \right) = ac\, e^{ax} = ay$$

Exercise 4. Does this solution also work if a is negative?

Let's change a to $-|a|$ in the result from the previous exercise:

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(c\, e^{-|a|x} \right) = -|a|\, c\, e^{ax} = -|a|\, y$$

So it still works fine.

Exercise 5. Using the method from the previous section, find this solution. Then verify it by testing the solution in the original differential equation.

Just as in the previous section, we have:

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= f(x)\,y \\
\frac{\mathrm{d}y}{y} &= \mathrm{d}x\,f(x) \\
\int \mathrm{d}y\,\frac{1}{y} &= \int \mathrm{d}x\,f(x) \\
\ln(y) &= \int \mathrm{d}x\,f(x) + c_1 \\
y &= c\,e^{\int \mathrm{d}x\,f(x)}
\end{aligned}
$$

The constant is added in to handle the constant of integration from the integral of $f(x)$.

Exercise 6. Verify that $y(x) = c\,e^{\sqrt{a}\,x}$ is a solution to $y'' = ay$. What about $y'' = -ay$? What is α such that $y(x) = c\,e^{\alpha x}$ is a solution to $y'' = -ay$?

To verify, once again we just plug and chug:

$$
\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(c\,e^{\sqrt{a}\,x}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{a}\,c\,e^{\sqrt{a}\,x}\right) \\
&= \sqrt{a}\sqrt{a}\,c\,e^{\sqrt{a}\,x} \\
&= ay
\end{aligned}
$$

In order to find α, we'll take the second derivative of the function:

$$y'' = \alpha^2 c\,e^{\alpha x} = \alpha^2 y$$

In order for this to satisfy the differential equation $y'' = -ay$, we must therefore have

$$\alpha^2 y = -ay \implies \alpha^2 = -a$$

Finally, we have:

$$\alpha = \sqrt{-a} = i\sqrt{a}$$

Exercise 7. Show that the solution $y(x) = c\,e^{\sqrt{-a}\,x}$ is not linearly indepedent of the solution $y(x) = A\cos(\sqrt{a}\,x) + B\sin(\sqrt{a}\,x)$. That is, find A and B such that the two solutions are equal.

Remembering Euler's formula, we have:

$$
\begin{aligned}
y(x) &= c\,e^{\sqrt{-a}\,x} \\
&= c\,e^{i\sqrt{a}\,x} \\
&= c\left(\cos(\sqrt{a}\,x) + i\sin(\sqrt{a}\,x)\right) \\
&= c\cos(\sqrt{a}\,x) + ic\sin(\sqrt{a}\,x)
\end{aligned}
$$

From that, we can identify $A = c$ and $B = ic$ in $A\cos(\sqrt{a}\,x) + B\sin(\sqrt{a}\,x)$.

Exercise 8. Verify that $y(x) = A\cos(\sqrt{a}\,x) + B\sin(\sqrt{a}\,x)$ is a solution to the differential equation $y'' = -ay$. Now what about $y'' = ay$? Find α such that $y(x) = A\cos(\alpha x) + B\cos(\alpha x)$ satisfies $y'' = ay$.

As always, to verify a solution to a differential equation, we just plug and chug.

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(A\cos(\sqrt{a}\,x) + B\sin(\sqrt{a}\,x)\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\sqrt{a}\,A\sin(\sqrt{a}\,x) + \sqrt{a}\,B\cos(\sqrt{a}\,x)\right)\\
&= \sqrt{a}\frac{\mathrm{d}}{\mathrm{d}x}\left(-A\sin(\sqrt{a}\,x) + B\cos(\sqrt{a}\,x)\right)\\
&= \sqrt{a}\left(-\sqrt{a}\,A\cos(\sqrt{a}\,x) - \sqrt{a}\,B\sin(\sqrt{a}\,x)\right)\\
&= -a\left(A\cos(\sqrt{a}\,x) + B\sin(\sqrt{a}\,x)\right)\\
&= -ay
\end{aligned}$$

Now we find α, again by taking the derivatives of the function:

$$y'' = -\alpha^2\left(A\cos(\alpha x) + B\sin(\alpha x)\right) = -\alpha^2 y$$

Setting this equal to the differential equation gives us $y'' = ay = -\alpha^2 y \implies a = -\alpha^2$. From there, we get

$$\alpha = \sqrt{-a} = i\sqrt{a}$$

Exercise 9. Suppose that we have the differential equation

$$\frac{-\hbar^2}{2m}\Psi'' = E\Psi$$

with the boundary conditions $\Psi(0) = \Psi(L) = 0$. What are allowed values for E? [Hint: this is the particle in a box. See exercise 2.]

We can quickly rearrange the equation above to give us the differential equation

$$\Psi'' = -\frac{2mE}{\hbar^2}\Psi$$

This in turn gives us the solution

$$\Psi(x) = A\cos\left(\sqrt{\frac{2mE}{\hbar^2}}x\right) + B\sin\left(\sqrt{\frac{2mE}{\hbar^2}}x\right)$$

From exercise 2 we know that the boundary values given will require that

$$\sqrt{\frac{2mE}{\hbar^2}} = \frac{n\pi}{L}$$

Solving for E, we obtain

$$E = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

Notice again that the quantization of energy levels comes about because of the boundary conditions, and the existence of multiple energy levels (the introduction of n) comes about because the wavefunction has multiple zeros (that is, there is more that one λ such that $\sin(\lambda L) = 0$).