Exercise 1.10

To derive Euler's theorem for nth-order homogeneous functions, we basically follow the same path that Chandler shows for the case of first-order functions.

First, we start by defining $u_i = \lambda x_i$ for all i. Then the definition of an nth order homogeneous function is, as given by Chandler's footnote on page 22,

$$f(\{u_i\}) = \lambda^n f(\{x_i\}) \tag{1}$$

where $i \in \{1, \ldots, N\}$.

Following Chandler, we'll take the λ-derivative of this in two different ways. First, we do the obvious and take the derivative of the right hand side.

$$\left(\frac{\partial f(\{u_i\})}{\partial \lambda}\right)_{\{x_i\}} = n\lambda^{n-1} f(\{x_i\}) \tag{2}$$

This is the analog of Chandler's equation (a).

Now we go from the other side: by knowing that each u_i is a function of λ, we can take the derivative using the chain rule:

$$\left(\frac{\partial f}{\partial \lambda}\right)_{\{x_i\}} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial u_i}\right)_{u_j} \left(\frac{\partial u_i}{\partial \lambda}\right)_{x_i} \tag{3}$$

$$= \sum_{i=1}^{N} \left(\frac{\partial f}{\partial u_i}\right)_{u_j} x_i \tag{4}$$

This, of course, is identical to Chandler's equation (b). (The left hand side of the definition of a homogeneous function, Eq. (1), is the same regardless of the degree n.)

Finally, we set these two things equal to each other. After a little rearrangement, that gives us:

$$f(\{x_i\}) = \frac{\lambda^{1-n}}{n} \sum_{i=1}^{N} \left(\frac{\partial f}{\partial u_i}\right)_{u_j} x_i \tag{5}$$

If, as Chandler did, we take $\lambda = 1$, we obtain for arbitrary-order homogeous functions:

$$f(\{x_i\}) = \frac{1}{n} \sum_{i=1}^{N} \left(\frac{\partial f}{\partial u_i}\right)_{u_j} x_i \tag{6}$$

In the case where $n = 1$, this obviously reduces the to result from Chandler.