Exercise 4.10

Let's start with Chandler's soon-to-be-profitable expansion for the energy,

$$\langle E \rangle = -\sum_{m=0}^{\infty} \frac{1}{m!} \Phi^{(m)}(\mu_0) \int_0^{\infty} \mathrm{d}\varepsilon \, \frac{\mathrm{d}F(\varepsilon)}{\mathrm{d}\varepsilon} (\varepsilon - \mu_0)^m \tag{1}$$

where I've used the parenthetical superscript of Φ as a derivative (for simplicity).

Now let's turn to the derivative of the Fermi function. We can easily do that analytically:

$$\frac{\mathrm{d}F(\varepsilon)}{\mathrm{d}\varepsilon} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \left(1 + e^{\beta(\varepsilon-\mu)}\right)^{-1} = \frac{\beta \, e^{\beta(\varepsilon-\mu)}}{(1 + e^{\beta(\varepsilon-\mu)})^2} \tag{2}$$

With that done, let's follow Chandler's suggestion and make the change of variables $x = \beta(\varepsilon - \mu)$. This gives us:

$$\mathrm{d}\varepsilon = \frac{\mathrm{d}x}{\beta} \tag{3a}$$

$$\varepsilon = \frac{x}{\beta} + \mu \tag{3b}$$

Finally, let's make a brief observation about the derivative of the Fermi function: as Chandler mentions (and shows in Fig. 4.3), it is highly localized around μ_0 — meaning that it is effectively zero everywhere that isn't within a couple $k_{\mathrm{B}}T$ of the Fermi energy. Since we know the Fermi energy is many $k_{\mathrm{B}}T$ above zero energy, we know that at $\varepsilon \leq 0$ we have $\mathrm{d}F(\varepsilon)/\mathrm{d}\varepsilon \approx 0$, and therefore the whole integrand is approximately zero. This means that we can extend the lower limit of integration to $-\infty$ without changing the results.

Using that observation and plugging Eqs. (2) and (3) into Eq. (1) gives us:

$$\langle E \rangle = -\sum_{m=0}^{\infty} \frac{1}{m!} \Phi^{(m)}(\mu_0) \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\beta} \, \frac{\beta \, e^{\beta(\frac{x}{\beta}+\mu-\mu)}}{(1 + e^{\beta(\frac{x}{\beta}+\mu-\mu)})^2} \left(\frac{x}{\beta} + \mu - \mu_0\right)^m \tag{4}$$

$$= -\sum_{m=0}^{\infty} \frac{1}{m!} \Phi^{(m)}(\mu_0) \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{e^x}{(1 + e^x)^2} \left(\frac{x}{\beta} + \mu - \mu_0\right)^m \tag{5}$$

If we now use the observation that μ and μ_0 are close at low temperature, this simplifies further:

$$\langle E \rangle = -\sum_{m=0}^{\infty} \frac{1}{m!} \Phi^{(m)}(\mu_0) \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{e^x}{(1 + e^x)^2} \left(\frac{x}{\beta}\right)^m \tag{6}$$

$$= -\sum_{m=0}^{\infty} (k_{\mathrm{B}}T)^m \frac{1}{m!} \Phi^{(m)}(\mu_0) \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{e^x}{(1 + e^x)^2} \, x^m \tag{7}$$

This, of course, gives us our expansion in terms of powers of $k_\mathrm{B}T$; the “constants” Chandler mentions are

$$(\text{constant})_m = \frac{1}{m!}\Phi^{(m)}(\mu_0)\int_{-\infty}^{\infty} \mathrm{d}x\, \frac{e^x}{(1+e^x)^2}\; x^m \tag{8}$$

But what happens to the odd powers? Well, that’s why we extended our integral to $-\infty$. There are a couple of ways to show that the integrand is an odd function for odd values of m (which results in the integral over all real numbers being zero). For example, you could go back to the Fermi function, show that it is an odd function shifted by constants in both dependent and independent variables, meaning that its derivative is even (shifted in the dependent variable). But I think it is a lot easier to just look at the functional form we have now. Define $g(x)$ as

$$g(x) \equiv \frac{e^x}{(1+e^x)^2} = \frac{e^x}{1+2e^x+e^{2x}} \tag{9}$$

$$= \frac{1}{e^{-x}+2+e^x} \tag{10}$$

It’s trivial to see that this is an even function; i.e., $\forall x$, $g(x) = g(-x)$. Multiply an even function by an odd function (any x^{2n+1} for integer n) and you get an odd function. So the integral is zero unless m is even. Our final result is:

$$\langle E \rangle = -\sum_{n=0}^{\infty}(k_\mathrm{B}T)^{2n}\left(\frac{1}{(2n)!}\Phi^{(2n)}(\mu_0)\int_{-\infty}^{\infty} \mathrm{d}x\, \frac{e^x}{(1+e^x)^2}\; x^{2n}\right) \tag{11}$$

which is just a more detailed way of writing Chandler’s result.